Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent’s path

- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put

- The agent receives rewards each time step
  - Small “living” reward each step (can be negative)
  - Big rewards come at the end (good or bad)

- Goal: maximize sum of (discounted) rewards
Recap: MDPs

- **Markov decision processes:**
  - States $S$
  - Actions $A$
  - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
  - Rewards $R(s,a,s')$ (and discount $\gamma$)
  - Start state $s_0$

- **Quantities:**
  - Policy = map of states to actions
  - Utility = sum of discounted rewards
  - Values = expected future utility from a state (max node)
  - Q-Values = expected future utility from a q-state (chance node)
Gridworld Values $V^*$

VALUES AFTER 100 ITERATIONS
Gridworld: $Q^*$

Q-VALUES AFTER 100 ITERATIONS
Optimal Quantities

- The value (utility) of a state \( s \):
  \[ V^*(s) = \text{expected utility starting in } s \text{ and acting optimally} \]

- The value (utility) of a q-state \( (s,a) \):
  \[ Q^*(s,a) = \text{expected utility starting out having taken action } a \text{ from state } s \text{ and (thereafter) acting optimally} \]

- The optimal policy:
  \[ \pi^*(s) = \text{optimal action from state } s \]
The Bellman Equations

How to be optimal:

Step 1: Take correct first action
Step 2: Keep being optimal
The Bellman Equations

- Definition of “optimal utility” via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values

\[ V^*(s) = \max_{a} Q^*(s, a) \]
\[ Q^*(s, a) = \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]
\[ V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]

- These are the Bellman equations, and they characterize optimal values in a way we’ll use over and over
**Value Iteration**

- Bellman equations characterize the optimal values:

\[ V^*(s) = \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^*(s') \right] \]

- Value iteration computes them:

\[ V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V_k(s') \right] \]

- Value iteration is just a fixed point solution method
  - ... though the \( V_k \) vectors are also interpretable as time-limited values
Convergence*

- How do we know the $V_k$ vectors are going to converge?

- Case 1: If the tree has maximum depth $M$, then $V_M$ holds the actual untruncated values

- Case 2: If the discount is less than 1
  - Sketch: For any state $V_k$ and $V_{k+1}$ can be viewed as depth $k+1$ expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, $V_{k+1}$ has actual rewards while $V_k$ has zeros
  - That last layer is at best all $R_{\text{MAX}}$
  - It is at worst $R_{\text{MIN}}$
  - But everything is discounted by $\gamma^k$ that far out
  - So $V_k$ and $V_{k+1}$ are at most $\gamma^k \max |R|$ different
  - So as $k$ increases, the values converge

\[ V_k(s) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad V_{k+1}(s) \]
Policy Methods
Fixed Policies

- Expectimax trees max over all actions to compute the optimal values.
- If we fixed some policy $\pi(s)$, then the tree would be simpler - only one action per state.
  - ... though the tree’s value would depend on which policy we fixed.
Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state $s$ under a fixed (generally non-optimal) policy

- Define the utility of a state $s$, under a fixed policy $\pi$:
  \[ V_\pi(s) = \text{expected total discounted rewards starting in } s \text{ and following } \pi \]

- Recursive relation (one-step look-ahead / Bellman equation):
  \[ V_\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_\pi(s')] \]
Example: Policy Evaluation

Always Go Right

Always Go Forward
Example: Policy Evaluation

**Always Go Right**

```
-10.00  100.00  -10.00
-10.00  1.09    -10.00
-10.00 -7.88    -10.00
-10.00 -8.69    -10.00
```

**Always Go Forward**

```
-10.00  100.00  -10.00
-10.00  70.20   -10.00
-10.00  48.74   -10.00
-10.00  33.30   -10.00
```
Policy Evaluation

- How do we calculate the V’s for a fixed policy $\pi$?

- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

- Efficiency: $O(S^2)$ per iteration

- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with Matlab (or your favorite linear system solver)
Policy Extraction
Computing Actions from Values

- Let’s imagine we have the optimal values $V^*(s)$
- How should we act?
  - It’s not obvious!
- We need to do a mini-expectimax (one step)

$$
\pi^*(s) = \arg\max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values
Computing Actions from Q-Values

- Let’s imagine we have the optimal q-values:

- How should we act?
  - Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$

- Important lesson: actions are easier to select from q-values than values!
Policy Iteration
Problems with Value Iteration

- Value iteration repeats the Bellman updates:
  \[ V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') \left( R(s, a, s') + \gamma V_k(s') \right) \]

- Problem 1: It’s slow - \( O(S^2A) \) per iteration

- Problem 2: The “max” at each state rarely changes

- Problem 3: The policy often converges long before the values

[Demo: value iteration (L9D2)]
$k=1$

VALUES AFTER 1 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k=2$

VALUES AFTER 2 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
<table>
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</tbody>
</table>

VALUES AFTER 3 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=4

VALUES AFTER 4 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=5

VALUES AFTER 5 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=7

VALUES AFTER 7 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k=8$

VALUES AFTER 8 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k = 9$

VALUES AFTER 9 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k=10

VALUES AFTER 10 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
$k=11$

VALUES AFTER 11 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
k = 12

VALUES AFTER 12 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0
\[ k = 100 \]

<p>| | | | |</p>
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<tr>
<td>0.49</td>
<td>0.43</td>
<td>0.48</td>
<td>0.28</td>
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VALUES AFTER 100 ITERATIONS

Noise = 0.2  
Discount = 0.9  
Living reward = 0
Policy Iteration

- Alternative approach for optimal values:
  - **Step 1: Policy evaluation:** calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - **Step 2: Policy improvement:** update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

- This is policy iteration
  - It’s still optimal!
  - Can converge (much) faster under some conditions
Policy Iteration

- **Evaluation**: For fixed current policy $\pi$, find values with policy evaluation:
  - Iterate until values converge:
    \[
    V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[ R(s, \pi_i(s), s') + \gamma V_{k}^{\pi_i}(s') \right]
    \]

- **Improvement**: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:
    \[
    \pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[ R(s, a, s') + \gamma V^{\pi_i}(s') \right]
    \]
Both value iteration and policy iteration compute the same thing (all optimal values)

In value iteration:
- Every iteration updates both the values and (implicitly) the policy
- We don’t track the policy, but taking the max over actions implicitly recomputes it

In policy iteration:
- We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
- After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
- The new policy will be better (or we’re done)

Both are dynamic programs for solving MDPs
Summary: MDP Algorithms

- So you want to....
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

- These all look the same!
  - They basically are - they are all variations of Bellman updates
  - They all use one-step lookahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions
Double Bandits
Double-Bandit MDP

- **Actions:** Blue, Red
- **States:** Win, Lose

No discount
100 time steps
Both states have the same value
Offline Planning

- **Solving MDPs is offline planning**
  - You determine all quantities through computation
  - You need to know the details of the MDP
  - You do not actually play the game!

<table>
<thead>
<tr>
<th>Value</th>
<th>Play Red</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Play Blue</td>
<td>100</td>
<td></td>
</tr>
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</table>

No discount
100 time steps
Both states have the same value
Let’s Play!
Online Planning

- Rules changed! Red’s win chance is different.
Let’s Play!

$0 \$0 \$0 \$2 \$0
$2 \$0 \$0 \$0 \$0
What Just Happened?

- That wasn’t planning, it was learning!
  - Specifically, reinforcement learning
  - There was an MDP, but you couldn’t solve it with just computation
  - You needed to actually act to figure it out

- Important ideas in reinforcement learning that came up
  - Exploration: you have to try unknown actions to get information
  - Exploitation: eventually, you have to use what you know
  - Regret: even if you learn intelligently, you make mistakes
  - Sampling: because of chance, you have to try things repeatedly
  - Difficulty: learning can be much harder than solving a known MDP
Next Time: Reinforcement Learning!