If utility function is $U(x) = x^2$, given a lottery $L = [0.5, 0; 0.5, 40]$

- Expected Utility: $EU(L) = ?$ (answer: 800)
  
  $EU(L) = \sum_{x \in L} p(x)U(x)$

- Equivalent Monetary Value is the amount of money you would pay in exchange for the lottery. $EMV(L) = ?$ (answer: $28.284$)
What Utilities to Use?

- For worst-case minimax reasoning, terminal function scale doesn’t matter
  - We just want better states to have higher evaluations (get the ordering right)
  - We call this **insensitivity to monotonic transformations**
What Utilities to Use?

- For worst-case minimax reasoning, terminal function scale doesn’t matter
  - We just want better states to have higher evaluations (get the ordering right)
  - We call this insensitivity to monotonic transformations

- For average-case expectimax reasoning, we need magnitudes to be meaningful
Hidden Markov Models

Instructor: Wei Xu
Ohio State University
[These slides were adapted from CS188 Intro to AI at UC Berkeley.]
Probability Recap

- Conditional probability
  \[ P(x|y) = \frac{P(x,y)}{P(y)} \]

- Product rule
  \[ P(x,y) = P(x|y)P(y) \]

- Chain rule
  \[ P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \]
  \[ = \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1}) \]

- X, Y independent if and only if: \( \forall x, y : P(x,y) = P(x)P(y) \) \( X \perp Y \)

- X and Y are conditionally independent given Z if and only if:
  \[ \forall x, y, z : P(x,y|z) = P(x|z)P(y|z) \]
  \( X \perp Y|Z \)
Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - \( P(\text{on time} \mid \text{no reported accidents}) = 0.90 \)
  - These represent the agent’s beliefs given the evidence

- Probabilities change with new evidence:
  - \( P(\text{on time} \mid \text{no accidents, 5 a.m.}) = 0.95 \)
  - \( P(\text{on time} \mid \text{no accidents, 5 a.m., raining}) = 0.80 \)
  - Observing new evidence causes beliefs to be updated
Inference in Ghostbusters

- A ghost is in the grid somewhere
- Sensor readings tell how close a square is to the ghost
  - On the ghost: red
  - 1 or 2 away: orange
  - 3 or 4 away: yellow
  - 5+ away: green

- Sensors are noisy, but we know $P(\text{Color} \mid \text{Distance})$

<table>
<thead>
<tr>
<th>$P(\text{red} \mid 3)$</th>
<th>$P(\text{orange} \mid 3)$</th>
<th>$P(\text{yellow} \mid 3)$</th>
<th>$P(\text{green} \mid 3)$</th>
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<td>0.15</td>
<td>0.5</td>
<td>0.3</td>
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</tbody>
</table>

[Demo: Ghostbuster - no probability (L12D1)]
Video of Demo Ghostbuster - No probability
Uncertainty

- General situation:
  - **Observed variables (evidence):** Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
  - **Unobserved variables:** Agent needs to reason about other aspects (e.g., where an object is or what disease is present)
  - **Model:** Agent knows something about how the known variables relate to the unknown variables

- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge
Video of Demo Ghostbusters with Probability
Hidden Markov Models
Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states $X$
  - You observe outputs (effects) at each time step
Example: Weather HMM

An HMM is defined by:

- Initial distribution: \( P(X_1) \)
- Transitions: \( P(X_t \mid X_{t-1}) \)
- Emissions: \( P(E_t \mid X_t) \)

<table>
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<tr>
<th>( R_t )</th>
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<td>+u</td>
<td>0.2</td>
</tr>
<tr>
<td>-r</td>
<td>-u</td>
<td>0.8</td>
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</table>
Example: Ghostbusters HMM

- $P(X_1) = \text{uniform}$
- $P(X|X') = \text{usually move clockwise, but sometimes move in a random direction or stay in place}$
- $P(R_{i,j}|X) = \text{same sensor model as before: red means close, green means far away.}$
Video of Demo Ghostbusters - Circular Dynamics -- HMM
Joint Distribution of an HMM

- Joint distribution:
  \[ P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3) \]

- More generally:
  \[ P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t) \]

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?
Chain Rule and HMMs

- From the chain rule, every joint distribution over \( X_1, E_1, X_2, E_2, X_3, E_3 \) can be written as:

\[
P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2)
\]

\[
P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)
\]

- Assuming that

\[
X_2 \perp E_1 \mid X_1, \quad E_2 \perp X_1, E_1 \mid X_2, \quad X_3 \perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp X_1, E_1, X_2, E_2 \mid X_3
\]

gives us the expression posited on the previous slide:

\[
P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)
\]
Chain Rule and HMMs

- From the chain rule, every joint distribution over $X_1, E_1, \ldots, X_T, E_T$ can be written as:

$$P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_1, E_1, \ldots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \ldots, X_{t-1}, E_{t-1}, X_t)$$

- Assuming that for all $t$:
  - State independent of all past states and all past evidence given the previous state, i.e.:

$$X_t \perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}$$

  - Evidence is independent of all past states and all past evidence given the current state, i.e.:

$$E_t \perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t$$

  gives us the expression posited on the earlier slide:

$$P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t)$$
Many implied conditional independencies, e.g.,

\[ E_1 \perp X_2, E_2, X_3, E_3 \mid X_1 \]

To prove them

- Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
- Approach 2: directly from the graph structure (3 lectures from now)
  - Intuition: If path between \( U \) and \( V \) goes through \( W \), then \( U \perp V \mid W \)
Real HMM Examples

- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options

- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
Filtering, or monitoring, is the task of tracking the distribution
\[ B_t(X) = P_t(X_t | e_1, ..., e_t) \] (the belief state) over time

- We start with \( B_1(X) \) in an initial setting, usually uniform

- As time passes, or we get observations, we update \( B(X) \)

- The Kalman filter was invented in the 60’s and first implemented as a method of trajectory estimation for the Apollo program
Example: Robot Localization

Example from Michael Pfeiffer

Sensor model: can read in which directions there is a wall, never more than 1 mistake
Motion model: may not execute action with small prob.
Example: Robot Localization

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake
Example: Robot Localization

$t=2$
Example: Robot Localization
Example: Robot Localization

$t=4$
Example: Robot Localization
Inference: Base Cases

\[ P(X_1|e_1) \]

\[
\begin{align*}
  P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\
  &\propto_{X_1} P(x_1, e_1) \\
  &= P(x_1)P(e_1|x_1)
\end{align*}
\]

\[ P(X_2) \]

\[
\begin{align*}
  P(x_2) &= \sum_{x_1} P(x_1, x_2) \\
  &= \sum_{x_1} P(x_1)P(x_2|x_1)
\end{align*}
\]
Assume we have current belief $P(X | \text{evidence to date})$

$$B(X_t) = P(X_t|e_{1:t})$$

Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step $t$ the belief is about, and what evidence it includes

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$
Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)
Assume we have current belief $P(X | \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1}|e_{1:t})}{P(e_{t+1}|e_{1:t})}$$

$$\propto P(X_{t+1}, e_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

Or, compactly:

$$B(X_{t+1}) \propto P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

- Basic idea: beliefs “rewighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize
Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

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Before observation

After observation

\[ B(X) \propto P(e|X)B'(X) \]
Example: Weather HMM

\[
\begin{align*}
B(+r) &= 0.5 \\
B(-r) &= 0.5 \\
B'(+r) &= 0.5 \\
B'(-r) &= 0.5 \\
B(+r) &= 0.818 \\
B(-r) &= 0.182 \\
B'(+r) &= 0.627 \\
B'(-r) &= 0.373 \\
B(+r) &= 0.883 \\
B(-r) &= 0.117
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
R_t & R_{t+1} & P(R_{t+1} | R_t) \\
\hline
+r & +r & 0.7 \\
+r & -r & 0.3 \\
-r & +r & 0.3 \\
-r & -r & 0.7 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
R_t & U_t & P(U_t | R_t) \\
\hline
+r & +u & 0.9 \\
+r & -u & 0.1 \\
-r & +u & 0.2 \\
-r & -u & 0.8 \\
\end{array}
\]
The Forward Algorithm

- We are given evidence at each time and want to know

\[ B_t(X) = P(X_t|e_{1:t}) \]

- We can derive the following updates

\[
P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})
= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})
= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)
= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})
\]

We can normalize as we go if we want to have \( P(x|e) \) at each time step, or just once at the end...
Online Belief Updates

- Every time step, we start with current $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t \mid e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} \mid e_{1:t-1}) \cdot P(x_t \mid x_{t-1})$$

- We update for evidence:

$$P(x_t \mid e_{1:t}) \propto \prod_{X} P(x_t \mid e_{1:t-1}) \cdot P(e_t \mid x_t)$$

- The forward algorithm does both at once (and doesn’t normalize)
Pacman - Sonar (P4)

[Demo: Pacman - Sonar - No Beliefs(L14D1)]
Video of Demo Pacman - Sonar (no beliefs)
Pacman - Sonar (P4)
Video of Demo Pacman - Sonar (with beliefs)
Next Time: Particle Filtering and Applications of HMMs