1 Joint and Conditional Distributions

Consider three random variables Toothache, Cavity and Catch. The joint probabilities that each random variable takes on the respective values is given below:

<table>
<thead>
<tr>
<th></th>
<th>+toothache</th>
<th>-toothache</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+catch</td>
<td>-catch</td>
</tr>
<tr>
<td>+cavity</td>
<td>0.108</td>
<td>0.012</td>
</tr>
<tr>
<td>-cavity</td>
<td>0.016</td>
<td>0.064</td>
</tr>
</tbody>
</table>

Questions:

1) What is \(P(+\text{toothache})\)?

\[ P(+\text{toothache}) = 0.108 + 0.016 + 0.012 + 0.064 = 0.2 \]

2) What is \(P(\text{Catch})\)?

\[ P(+\text{catch}) = 0.108 + 0.016 + 0.072 + 0.144 = 0.34 \]
\[ P(-\text{catch}) = 1 - P(+\text{catch}) = 0.66 \]

3) What is \(P(+\text{cavity} | +\text{catch})?\)

\[ P(+\text{cavity} | +\text{catch}) = \frac{P(+\text{cavity}, +\text{catch})}{P(+\text{catch})} = \frac{0.108 + 0.072}{0.34} \approx 0.53 \]

4) What is \(P(+\text{cavity} | +\text{toothache or +catch})?\)

\[ P(+\text{cavity} | +\text{toothache or +catch}) = \frac{P(+\text{cavity}, +\text{toothache or +catch})}{P(+\text{toothache or +catch})} = \frac{0.108 + 0.012 + 0.072}{0.108 + 0.012 + 0.072 + 0.016 + 0.064 + 0.144} \approx \frac{0.192}{0.416} \approx 0.46 \]

5) Is the random variable Catch conditionally independent of Toothache, given Cavity?

(Hint: \(P(+\text{catch} | +\text{toothache}, +\text{cavity}) = P(+\text{catch} | +\text{cavity}) \)?)

Yes. Because

\[ P(+\text{catch} | +\text{toothache}, +\text{cavity}) = \frac{P(+\text{catch}, +\text{toothache}, +\text{cavity})}{P(+\text{toothache}, +\text{cavity})} = \frac{0.108}{0.08 + 0.012} = 0.9 \]

\[ P(+\text{catch} | +\text{cavity}) = \frac{P(+\text{catch}, +\text{cavity})}{P(+\text{cavity})} = \frac{0.108 + 0.072}{0.08 + 0.072 + 0.012 + 0.064} = \frac{0.18}{0.29} \approx 0.9 \]
2 Conditional Independence

For random variables $X, Y, Z$, show that the following three statements are equivalent:

(i) $P(X|Y, Z) = P(X|Z)$
(ii) $P(Y|X, Z) = P(Y|Z)$
(iii) $P(X, Y|Z) = P(X|Z)P(Y|Z)$

Equivalence of the first two statements show that conditional independence is symmetric ($X$ and $Y$ are conditionally independent given $Z$, and the order of $X$ and $Y$ doesn't matter). The third statement is analogous to the definition of unconditional independence: $P(X, Y) = P(X)P(Y)$.

Showing equivalence of (i) and (iii)

$P(X, Y|Z) = P(X|Z)P(Y|Z)$

$\downarrow$

$P(X|Y, Z) = P(X|Z)$

plug-in

$P(X|Y, Z) = \frac{P(X, Y, Z)}{P(Y, Z)} = \frac{P(X, Y|Z)P(Z)}{P(Y, Z)}$

$= \frac{P(X|Z)P(Y|Z)P(Z)}{P(Y, Z)}$